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On the number operators of single-mode q-oscillators

R Chakrabarti† and R Jagannathan‡

† Department of Theoretical Physics, University of Madras, Guindy Campus, Madras-600 025, India
‡ The Institute of Mathematical Sciences, C.I.T. Campus, Tharamani, Madras-600 113, India

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Abstract. This paper presents the available solutions to the problem of expressing the number operator in terms of the creation and annihilation operators in the case of the various single-mode q-oscillators. This study reveals interesting number theoretic aspects of the problem.

Recently it has been shown [1, 2] that for a single-mode q-oscillator defined by the relations

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \tag{1a}$$

$$aa^\dagger - qa^\dagger a = 1 \tag{1b}$$

with real $q \in [-1, \infty)$, the excitation number operator N can be expressed in terms of the creation ($a^\dagger$) and the annihilation (a) operators as

$$N = \sum_{m=1}^{\infty} \frac{(1-q)^m}{(1-q^m)} (a^\dagger)^m a^m. \tag{2}$$

The spectrum of N has been assumed to be $(0, 1, 2, \ldots)$ and the ground state $|0\rangle$ ($N|0\rangle = 0$, $a|0\rangle = 0$) is taken to be non-degenerate. It is straightforward to derive the expression (2) by substituting the ansatz

$$N = \sum_{m=1}^{\infty} \nu_m (a^\dagger)^m a^m \tag{3}$$

in (1a) and using (1b) to obtain the coefficients $\{\nu_m\}$ recursively. In this paper we obtain N in the same form as in (3) for other kinds of deformed oscillators different from (1). In general, we shall take a deformed oscillator algebra to be defined by the relations (1a) and

$$aa^\dagger - \xi a^\dagger a = \mu(N). \tag{4}$$

ξ and $\mu(N)$ should be such that the function

$$h(n) = \sum_{k=0}^{n-1} \xi^k \mu(n-k-1) \qquad n \geqslant 1,\ h(0) = 0 \tag{5}$$

is real and non-negative. This study reveals interesting number theoretic aspects of the problem of expressing N in terms of $(a, a^\dagger)$ in the case of deformed oscillator algebras. For certain special choices of ξ and $\mu(N)$ other types of expression for N in terms of $(a, a^\dagger)$, different from (3), are also possible as we shall see below.

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Generalized oscillators of the type (1) have been the subject of study in different contexts: dual resonance models in high energy physics [3], generalized coherent states [4], exploration of new methods of quantization [5–8], and new forms of quantum statistics [9–11]. The emergence of quantum algebras, as common structures underlying several mathematical and physical theories, has led to the development of the q-boson oscillator algebra [12–17] with

$$aa^{\dagger} - qa^{\dagger}a = q^{-N}. \tag{6}$$

The fermionic counterpart of (6), the q-fermion oscillator algebra, with

$$aa^{\dagger} + qa^{\dagger}a = q^{-N} \tag{7}$$

has also been studied [17, 18] in connection with quantum superalgebras. The boson and the fermion oscillators are obtained as the $q = 1$ limit of (6) and (7) respectively. In the case of the algebra (1) both the boson and the fermion algebras are obtained as special cases in the limits $q = 1$ and $q = -1$ respectively. Another q-oscillator with

$$aa^{\dagger} - qa^{\dagger}a = q^{-2N} \tag{8}$$

has been studied in [10]; this algebra (8) also has both the boson and the fermon algebras as limiting cases corresponding to $q = 1$ and $q = -1$ respectively.

As a generalization of the deformed oscillator algebras (1, 6–8) we introduced in [19] the (p, q)-oscillator with

$$aa^{\dagger} - qa^{\dagger}a = p^{-N}. \tag{9}$$

As shown by us in detail in [19], through several examples, the (p, q)-oscillator algebra (9) provides a convenient language for studying the two-parameter (p, q) extensions of the standard quantum algebras with a single deformation parameter (q). The (p, q)-oscillator algebra (9) has also been noted in [20], as the two-parameter generalization of the standard ($su_q(2)$-related) q-oscillator algebra (6). In [20] the study is, primarily, to obtain the time-evolution of such deformed oscillator systems using the formalism of Lie-admissible algebras.

The general deformed oscillator algebra (4) has been used in [21] to provide a unified picture of all the known (that is, standard (usual boson, fermion, parafermion and paraboson) and single-parameter (q)) quantizations of the simple harmonic oscillator (see also [20, 22, 23]). The two-parameter oscillator (9) is also a special case of (4). Further, it is noted in [21] that the general structure (4) may also be used to construct new deformations of the oscillator algebra; for example, a new oscillator called the Tamm–Dancoff cut-off q-oscillator is introduced corresponding to the choice $\xi = q$, $\mu(N) = q^N$. This Tamm–Dancoff cut-off q-oscillator may also be considered as the limiting case of (9) with $p = q^{-1}$. To build a physical theory based on these deformed algebras it is necessary to express N in terms of $(a, a^{\dagger})$ and the first step is the treatment of the single-mode case.

Let us now consider the construction of N in the case of the general deformed oscillator (4). The commutation relations (4) imply that, up to phase factors, the orthonormal eigenstates of N, $\{|n\rangle \,|\, N|n\rangle = n|n\rangle,\ n = 0, 1, 2, \ldots\}$, are given by

$$\sqrt{h(n)}\,|n\rangle = a^{\dagger}|n-1\rangle \qquad n = 1, 2, 3, \ldots \tag{10}$$

where $h(n)$ is defined, uniquely, by (5) such that

$$h(n+1) - \xi h(n) = \mu(n) \qquad n = 0, 1, 2, \ldots. \tag{11}$$

In the space of the Fock states $\{|n\rangle\}$,

$$|\langle m|a|n\rangle|^2 = h(n)\delta_{n,m+1} \qquad \langle m|a^\dagger|n\rangle = \langle n|a|m\rangle^* \qquad m, n = 0, 1, 2, \ldots . \tag{12}$$

With this matrix realization (12), the coefficients $\{\nu_m\}$ in the expression (3) for N are seen to obey the relations

$$\sum_{m=1}^{n} \frac{\nu_m}{h(n-m)!} = \frac{n}{h(n)!} \qquad n = 1, 2, \ldots$$
$$h(n)! = h(n)h(n-1)\ldots h(2)h(1) \qquad h(0)! = 1. \tag{13}$$

The unique solution for $\{\nu_m\}$ is obtained by solving (13) recursively. The result is

$$\begin{aligned}\nu_m &= \frac{m}{h(m)!} + \frac{(m-1)}{h(m-1)!}\left\{-\frac{1}{h(1)!}\right\} + \frac{(m-2)}{h(m-2)!}\left\{-\frac{1}{h(2)!} + \frac{1}{h(1)!h(1)!}\right\} \\ &\quad + \frac{(m-3)}{h(m-3)!}\left\{-\frac{1}{h(3)!} + \frac{1}{h(1)!h(2)!} + \frac{1}{h(2)!h(1)!} \right. \\ &\quad \left. - \frac{1}{h(1)!h(1)!h(1)!}\right\} + \ldots + \frac{1}{h(1)!}\left\{-\frac{1}{h(m-1)!} + \ldots\right\} \\ &= \frac{m}{h(m)!} + \sum_{k=1}^{m-1} \frac{(m-k)}{h(m-k)!}\left(\sum_{\substack{c_1+c_2+\ldots+c_p=k \\ c_j>0}} \frac{(-1)^p}{h(c_1)!h(c_2)!\ldots h(c_p)!}\right).\end{aligned} \tag{14}$$

Thus, the equations (3) and (14) define N in terms of $(a, a^\dagger)$ in the case of the general deformed oscillator algebra (4).

Now, for the (p, q)-oscillator algebra (9) $h(n) = (q^n - p^{-n})/(q - p^{-1})$, p and q can be real, or $p^*q = 1$ for complex p and q, subject to the condition that this $h(n)$ is non-negative. For the q-oscillator (1), corresponding to $p = 1$, q has to be real ($q \in [-1, \infty)$), and $h(n) = (q^n - 1)/(q-1) = [n]$. Comparing the results (2) and (14) in this case one has the identity

$$\frac{(1-q)^m}{(1-q^m)} = \frac{m}{[m]!} + \sum_{k=1}^{m-1} \frac{(m-k)}{[m-k]!}\left(\sum_{\substack{c_1+c_2+\ldots+c_p=k \\ c_j>0}} \frac{(-1)^p}{([c_1]![c_2]!\ldots[c_p]!)}\right). \tag{15}$$

The q-deformed number $[n]$, referred to as the *basic number* in the number theory literature, was introduced originally the Heine [24]. Using the notation of number theory the identity (15) can be written as

$$\frac{1}{(1-q^m)} = \frac{m}{(q;q)_m} + \sum_{k=1}^{m-1} \frac{(m-k)}{(q;q)_{m-k}}\left(\sum_{\substack{c_1+c_2+\ldots+c_p=k \\ c_j>0}} \frac{(-1)^p}{((q;q)_{c_1}\ldots(q;q)_{c_p})}\right) \tag{16}$$

where

$$(q;q)_n = (1-q)(1-q^2)\ldots(1-q^n) = (1-q)^n([n]!). \tag{17}$$

The identity (15) has also been noted in [2] in a slightly different form. In the case of the usual boson corresponding to the limit $p = 1$ and $q = 1$, $h(n) = n$, $\nu_m = \delta_{m1}$, and the identity (15) reduces to

$$\frac{1}{(m-1)!} + \sum_{k=1}^{m-1} \frac{1}{(m-k-1)!}\left(\sum_{\substack{c_1+c_2+\ldots+c_p=k \\ c_j>0}} \frac{(-1)^p}{(c_1!c_2!\ldots c_p!)}\right) = \delta_{m1}. \tag{18}$$

For an oscillator with $q=0$ and $p=1$ (see [9]) $h(n)=1$ for any $n \geqslant 1$ and $\nu_m=1$ for all m. Correspondingly, the identity (15) becomes

$$m+\sum_{k=1}^{m-1}(m-k)\left\{\sum_{p=1}^{k}(-1)^p\binom{k-1}{p-1}\right\}=1 \qquad \forall m \geqslant 1. \tag{19}$$

Here, we have used the result [25] that the number of compositions (partitions considering also the order of the summands) of k with exactly p parts is $\binom{k-1}{p-1}$. It would be worthwhile to see whether number theory can help to reduce the sum on the right-hand side of (14) to a more closed form for generic values of p and q.

If p and q are complex and $(pq)^{1/2}$ is an mth primitive root of unity then $h(n)$ becomes zero at $n=m$ for odd m and at $n=m/2$ for even m. Then the corresponding oscillator system has only m (for odd m) or $m/2$ (for even m) states in view of equation (10), and expression (3) for N contains only a finite number of terms. This has been noted in [1] in the case of the usual fermion ($p=1, q=-1$).

Besides expression (3) one can write N in terms of $(a, a^\dagger)$ in other ways also for particular cases; such expressions may not be valid for all admissible values of p and q. For example, in the case of the oscillator (1) with $q>0$

$$N=\ln\{1-(1-q)a^\dagger a\}/\ln q \tag{20}$$

as noted in [10]. In general, when $p=q^r$, with q (real) >0 and r as any integer $\geqslant 0$, we have

$$N=\ln\{1-(q^{-r}-q)(aa^\dagger-q^{-r}a^\dagger a)^r a^\dagger a\}/((r+1)\ln q). \tag{21}$$

To see this, one has to observe from the matrix realization (12) that for the (p, q)-oscillator, in general,

$$aa^\dagger-p^{-1}a^\dagger a=q^N$$
$$a^\dagger a=h(N)=(q^N-p^{-N})/(q-p^{-1}). \tag{22}$$

The oscillators (1), (6) and (8) correspond respectively to $r=0$, 1 and 2. Another expression for N in this case is

$$N=\sum_{m=1}^{\infty}\frac{(1-q^{r+1})^m}{(1-q^{m(r+1)})}q^{-rm(m+1)/2}(a^\dagger)^m a^m(aa^\dagger-q^{-r}a^\dagger a)^{rm}. \tag{23}$$

This result is obtained by replacing in (2) q by q^{r+1} and a by $(q^{rN/2}a)$ and using the relations

$$q^{rN/2}aa^\dagger q^{rN/2}-q^{r+1}a^\dagger q^{rN}a=1 \qquad aa^\dagger-q^{-r}a^\dagger a=q^N. \tag{24}$$

Let us now make a few remarks on expression (3) for N written in the form

$$N=a^\dagger A=A^\dagger a \tag{25}$$

following [1]. Generalizing the observation in [1] in the case of N given by (2) for the oscillator (1), one has

$$[A, a^\dagger]=[a, A^\dagger]=1 \tag{26}$$

and hence the coherent states of the corresponding oscillators may be obtained by defining $|z\rangle \sim \{\exp(zA^\dagger)\}|0\rangle$. There is another interesting application of the result (2) in connection with the q-analysis [26–28]. Defining the q-derivative (or the q-difference) operator

$$D_q\psi(x) = \frac{\psi(qx)-\psi(x)}{qx-x} = \frac{(q^{x\,d/dx}-1)\psi(x)}{(q-1)x} \tag{27}$$

following Jackson [26], one has a realization of the relations (1) in the identification

$$a = D_q \qquad a^\dagger = x \qquad N = x\frac{d}{dx}. \tag{28}$$

Now the relations $N = a^\dagger A$ and $[A, a^\dagger] = 1$ imply that we must identify A with d/dx in this realization. Thus one has the identity

$$\frac{d\psi}{dx} = \sum_{m=1}^{\infty} \frac{(1-q)^m}{(1-q^m)} x^{m-1}(D_q)^m\psi \tag{29}$$

as the inverse relation for (27); this result may be of relevance for numerical analysis.

Finally, to conclude, let us make the observation that the $(a, a^\dagger)$ commutation relations of the (p, q)-oscillator, namely $aa^\dagger - qa^\dagger a = p^{-N}$ and $aa^\dagger - p^{-1}a^\dagger a = q^N$, may be written in an N-independent form as

$$[a, [a, a^\dagger]_q]_{p^{-1}} = [a, [a, a^\dagger]_{p^{-1}}]_q = 0 \tag{30}$$

with the notation $[A, B]_\alpha = AB - \alpha BA$. Written in the form (30), the $q \leftrightarrow p^{-1}$ symmetry of the (p, q)-oscillator is manifest. Furthermore, it may be noted that in view of the relation $[a, N] = a$, or $a\alpha^N - \alpha^{N+1}a = 0$ for any α, the $(a, a^\dagger)$ commutation relations follow uniquely from (30).

Note added in proof. For multimode systems of deformed oscillators covariant under several quantum groups the number operators have been constructed in [29].

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